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Trapped interacting two-component bosons in one dimension

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Abstract

In this paper we solve one-dimensional trapped $SU(2)$ bosons with repulsive δ-function interaction by means of the Bethe-ansatz method. The features of ground state and low-lying excited states are studied by numerical and analytic methods. We show that the ground state is an isospin 'ferromagnetic' state which differs from spin-$\frac{1}{2}$ fermion systems. There exist three quasi-particles in the excitation spectra, and both holon–antiholon and holon–isospinon excitations are gapless for large systems. The equilibrium thermodynamics of the system at finite temperature is studied using the thermodynamic Bethe ansatz. The thermodynamic quantities, such as specific heat, etc, are obtained for the case of the strong coupling limit.

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1. Introduction

In the past few years, Bose–Einstein condensation (BEC) has witnessed a sequence of exhilarating experimental achievements, and much attention has been paid to the study of Bose systems. The possibility of BEC in one dimension has been discussed for the non-interacting Bose gas [1, 2]. The role of dimensionality has been examined for the ideal Bose gas [3]. It is known that the interaction between bosons plays an essential role in one dimension due to the strong constraint in phase space [4]. The Luttinger liquid properties [5] of a trapped, interacting quasi-one-dimensional Bose gas have been discussed [6]. It was shown that under suitable experimental conditions the system can be described as a Luttinger liquid and the correlation function of the bosons decays algebraically which prevents Bose–Einstein condensation. Recently, a two-component Bose gas was produced in magnetically trapped ^{87}Rb by rotating the two hyperfine states into each other with the help of a slightly detuned Rabi oscillation field [7, 8]. It was noticed [9] that the properties of the Bose system can be different from the traditional scalar Bose system once it acquires internal degrees of freedom.

The two-component Bose system on a circle (periodic boundary conditions) has been studied by means of the Bethe-ansatz method [10]. It was shown that the ground state is an isospin-ferromagnetic state which differs from the one-dimensional spin-$\frac{1}{2}$ Fermi system whose ground state is the $SU(2)$ singlet.

In this paper, we solve trapped two-component bosons with δ-function interaction in one dimension using the Bethe-ansatz method. On the basis of the Bethe-ansatz equation, we discuss the ground state, low-lying excited states and the thermodynamics of the system at finite temperature, where some thermal quantities at low temperature are obtained explicitly. Our paper is organized as follows: in the next section we introduce the model and derive a set of nonlinear equations for charge rapidity and isospin rapidity. In section 3, we explicitly show that the ground state is an isospin 'ferromagnetic' state and manifest how the quantum numbers in the Bethe-ansatz equation should be taken for the ground state. In section 4, we study the low-lying excited states extensively by analysing the possible variations in the sequence of quantum numbers. Numerical results of energy–'momentum' spectra for each excitation are given. In section 5 we discuss the general thermodynamics of the system by means of the thermodynamic Bethe ansatz [13]. In section 6 we derive the free energy and specific heat in the case of the strong coupling limit. In the last section a brief summary is given.

2. The model and its Bethe-ansatz solution

The Hamiltonian for two-component bosons trapped in a potential well of infinite depth reads

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N} V(x_i) + 2c \sum_{i>j=1}^{N} \delta(x_i - x_j) \tag{1}$$

where

$$V(x_i) = \begin{cases} 0 & |x| \leqslant L/2 \\ \infty & |x| > L/2. \end{cases} \tag{2}$$

The Hamiltonian is not invariant under translation due to the presence of a trapping potential, and the total momentum of the system is not conserved. However, the system is still invariant under the action of the permutation group S_N which makes it possible to employ the coordinate Bethe-ansatz approach.

In the domain with $x_i \neq x_j\,(i \neq j)$ and inside the potential well, the Hamiltonian reduces to the one for free bosons and its eigenfunctions are therefore just superpositions of plane waves. When two particles collide with each other at the same point, a scattering process occurs. The Bethe-ansatz embodies that this process is purely elastic and the occurrence during the process is merely that the particles exchange their momenta. So for a given momentum $k = (k_1, k_2, \ldots, k_N)$, the scattering momenta include all permutations of the components of k. Moreover, since the total momentum is not conserved in the present model, we will have more nondiffractive scattering momenta, e.g. for $N = 2$, we have $(k_1, k_2) \to (-k_1, k_2)$, $(k_1, -k_2)$, $(-k_1, -k_2)$, (k_2, k_1), $(-k_2, k_1)$, $(k_2, -k_1)$, or $(-k_2, -k_1)$. All eight states correspond to the same energy $k_1^2 + k_2^2$. Hence, besides all possible permutations of the components of k, the scattering must include all possibilities of sign changes in the components of k.

Now we consider the case of N bosons. Considering the fact that the total momentum is not conserved, we adopt the following Bethe-ansatz form [14]:

$$\psi_a(x) = \sum_{P \in \mathcal{W}_B} A_a(P, Q) \mathrm{e}^{\mathrm{i}(Pk|Qx)} \qquad x \in \mathcal{C}(Q) \tag{3}$$

where $a = (a_1, a_2, \ldots, a_N)$, a_j denotes the isospin label of the jth particle; Pk stands for the image of a given $k := (k_1, k_2, \ldots, k_N)$ by a mapping $P \in \mathcal{W}_B$ and the coefficients $A(P, Q)$

are functionals on $\mathcal{W}_B \otimes \mathcal{W}_A$. Here, $\mathcal{W}_B$ and $\mathcal{W}_A$ stand for the Weyl groups of the B_N and A_{N-1} Lie algebras respectively. The later is isomorphic to the permutation group S_N and the former consists of S_N and all possible sign changes. We emphasize that the sum runs over the Weyl group of Lie algebra B_N but the wavefunction is defined on various Weyl chambers $\mathcal{C}(Q)$ corresponding to the Weyl group of A_{N-1} Lie algebra. This is different to the situation of periodic boundary conditions.

For a Bose system, the wavefunction is supposed to be symmetric under any permutation of both coordinates and isospin indices, i.e.

$$(\sigma^j \psi)_a(x) = \psi_a(x). \tag{4}$$

Here $(\sigma^j \psi)_a$ is well defined by $\psi_{\sigma^j a}(\sigma^j x)$, therefore both sides of (4) can be written out by using (3). Furthermore, using the evident identity $(Pk|\sigma^i Qx) = (\sigma^i Pk|Qx)$ and the rearrangement theorem of group theory, we obtain the following consequence from (4):

$$A_a(P, \sigma^i Q) = A_{\sigma^j a}(\sigma^i P, Q). \tag{5}$$

The δ-function term in the Hamiltonian (1) which contributes boundary conditions at the hyperplane P_α (α is a root of Lie algebra A_{N-1}), namely a discontinuity of the derivatives of the wavefunction along the normal of a Weyl hyperplane:

$$\lim_{\epsilon \to 0^+} [\alpha \cdot \nabla \psi_a(x_{(\alpha)} + \epsilon\alpha) - \alpha \cdot \nabla \psi_a(x_{(\alpha)} - \epsilon\alpha)] = 2c\psi_a(x_{(\alpha)}) \tag{6}$$

where $x_{(\alpha)} \in \mathsf{P}_\alpha$ and $\nabla := \sum_{i=1}^N e_i(\partial/\partial x_i)$. Substituting (3) into (6), we find that

$$\begin{aligned} &\mathrm{i}[(Pk)_j - (Pk)_{j+1}][A_a(P, \sigma_i Q) - A_a(\sigma_i P, \sigma_i Q) - A_a(P, Q) + A_a(\sigma_i P, Q)] \\ &\quad = c[A_a(P, Q) + A_a(\sigma_i P, Q) + A_a(P, Q) + A_a(\sigma_i P, Q)]. \end{aligned} \tag{7}$$

By making use of the relation (5) and the continuity relation of the wavefunction across $(Qx)_i = (Qx)_{i+1}$, we can obtain the following relations:

$$A_a(\sigma_i P, Q) = \check{S}^i_{a,a'}(Pk) A_{a'}(P, Q) \tag{8}$$

$$\check{S}^i_{a,a'}(Pk) = -\frac{c\delta_{a,a'} + \mathrm{i}[(Pk)_i - (Pk)_{i+1}]\mathcal{P}_{a,a'}}{c - \mathrm{i}[(Pk)_i - (Pk)_{i+1}]} \tag{9}$$

where $\mathcal{P}_{a,a'}$ stands for the matrix elements of the spinor representation of the permutation group. The relation (4) provides for the A coefficients a relation between different Weyl chambers. Equation (8) provides a connection between those coefficients which are related via any element of the Weyl group $\mathcal{W}_B$ in the same Weyl chamber.

The basic elements of the Weyl group $\mathcal{W}_B$ obey $\sigma^i\sigma^i = 1$ and $\sigma^i\sigma^{i+1}\sigma^i = \sigma^{i+1}\sigma^i\sigma^{i+1}$ as identities. These identities imply that $A_a(\sigma^i\sigma^i P, Q) = A_a(P, Q)$ and $A_a(\sigma^i\sigma^{i+1}\sigma^i P, Q) = A_a(\sigma^{i+1}\sigma^i\sigma^{i+1} P, Q)$. Using (8) repeatedly, one can obtain the following relations:

$$\begin{aligned} &\check{S}^i(\sigma^i Pk)\check{S}^i(\sigma k) = I \\ &\check{S}^i(\sigma^{i+1}\sigma^i Pk)\check{S}^{i+1}(\sigma^i Pk)\check{S}^i(Pk) = \check{S}^{i+1}(\sigma^i\sigma^{i+1} Pk)\check{S}^i(\sigma^{i+1} Pk)\check{S}^{i+1}(Pk) \end{aligned} \tag{10}$$

where we have adopted the conventions $\check{S} = \mathrm{mat}(S_{ab})$, $\check{S}^i = \check{S} \otimes I$, $\check{S}^{i+1} = I \otimes \check{S}$ (I is a 2×2 unit matrix). These relations are consistency conditions for the S-matrix. The second relation is called the Yang–Baxter equation. The concrete S-matrix in (9) fulfils these relations.

Due to the infinite depth of the potential well, the wavefunction must vanish at the ends of the well:

$$\psi_a((Qx)_1 = -L/2) = 0 \qquad \psi_a((Qx)_N = L/2) = 0. \tag{11}$$

These boundary conditions give rise to

$$A_a(\tau^1 P, Q) = -\mathrm{e}^{-\mathrm{i}(Pk)_1 L} A_a(P, Q) \tag{12}$$

$$A_a(\gamma P, Q) = -\mathrm{e}^{\mathrm{i}(Pk)_N L} A_a(P, Q) \tag{13}$$

where $\gamma = \sigma^{N-1} \ldots \sigma^1 \tau^1 \sigma^1 \ldots \sigma^{N-1}$. The relation (12) together with equation (8) determine the A amplitudes up to an overall factor since $\sigma^1, \ldots, \sigma^{N-1}$ and τ^1 generate the whole $\mathcal{W}_B$. There are two further consistency conditions. One comes from the identity ($\tau^1\sigma^1\tau^1\sigma^1 = \sigma^1\tau^1\sigma^1\tau^1$) of the Weyl group $\mathcal{W}_B$ leading to a reflection-related Yang–Baxter equation which is fulfilled automatically due to the fact that the reflection matrix is just the unit matrix multiplied by a scalar function. The other one comes from equation (13) which leads to an eigenvalue equation for the products of the S-matrices. This eigen-equation can be diagonalized by means of a standard quantum inverse scattering method [15]. One finally obtains the following Bethe-ansatz equations:

$$\begin{aligned} \mathrm{e}^{\mathrm{i}2k_jL} &= \prod_{l\neq j}^{N} \frac{k_j - k_l + \mathrm{i}c}{k_j - k_l - \mathrm{i}c} \frac{k_j + k_l + \mathrm{i}c}{k_j + k_l - \mathrm{i}c} \prod_{\mu=1}^{M} \frac{k_j - \lambda_\mu - \mathrm{i}c/2}{k_j - \lambda_\mu + \mathrm{i}c/2} \frac{k_j + \lambda_\mu - \mathrm{i}c/2}{k_j + \lambda_\mu + \mathrm{i}c/2} \\ 1 &= \prod_{l=1}^{N} \frac{\lambda_\gamma - k_l - \mathrm{i}c/2}{\lambda_\gamma - k_l + \mathrm{i}c/2} \frac{\lambda_\gamma + k_l - \mathrm{i}c/2}{\lambda_\gamma + k_l + \mathrm{i}c/2} \prod_{\mu\neq\gamma}^{M} \frac{\lambda_\gamma - \lambda_\mu + \mathrm{i}c}{\lambda_\gamma - \lambda_\mu - \mathrm{i}c} \frac{\lambda_\gamma + \lambda_\mu + \mathrm{i}c}{\lambda_\gamma + \lambda_\mu - \mathrm{i}c} \end{aligned} \tag{14}$$

where M denotes the total number of down isospins and λ denote isospin rapidities which arise from the diagonalization conditions of the quantum inverse scattering method. Taking the logarithm of equations (14) gives rise to the secular equations

$$\begin{aligned} \pi I_j &= k_j L + \sum_{l\neq j}^{N} \left[\tan^{-1}\left(\frac{k_j - k_l}{c}\right) + \tan^{-1}\left(\frac{k_j + k_l}{c}\right)\right] \\ &\quad - \sum_{\mu=1}^{M} \left[\tan^{-1}\left(\frac{k_j - \lambda_\mu}{c/2}\right) + \tan^{-1}\left(\frac{k_j + \lambda_\mu}{c/2}\right)\right] \\ \pi J_\gamma &= \sum_{l=1}^{N} \left[\tan^{-1}\left(\frac{\lambda_\gamma - k_l}{c/2}\right) + \tan^{-1}\left(\frac{\lambda_\gamma + k_l}{c/2}\right)\right] \\ &\quad - \sum_{\mu\neq\gamma}^{M} \left[\tan^{-1}\left(\frac{\lambda_\gamma - \lambda_\mu}{c}\right) + \tan^{-1}\left(\frac{\lambda_\gamma + \lambda_\mu}{c}\right)\right] \end{aligned} \tag{15}$$

where I_j is the quantum number for the charge rapidity k_j and J_γ for the isospin rapidity. Concerning the property of the logarithm function, I_j and J_γ take integer values regardless of either the total number of particles or that of isospin down. Once the roots are solved from equations (15) for a given set of quantum numbers $\{I_j, J_\gamma\}$, the energy can be calculated using $E = \sum_{j=1}^{N} k_j^2$.

3. The ground state

For any set of quantum numbers $\{I_j, J_\gamma\}$ with the solution $\{k_j, \lambda_\gamma\}$, the replacement of either $I_j \to -I_j, k_j \to -k_j$ or $J_\gamma \to -J_\gamma, \lambda_\gamma \to -\lambda_\gamma$ keeps the secular equation (15) invariant. So it makes no change to the energy, then we only need to consider the case of positive integer [16] for Is and Js. In the weak coupling limit $c \to 0$, $\tan^{-1}(x/c) \to \pi\,\mathrm{sgn}(x)/2$, equations (15) become

$$\begin{aligned} \pi I_j &= k_j L + \frac{\pi}{2} \sum_{l\neq j}^{N} [\mathrm{sgn}(k_j - k_l) + \mathrm{sgn}(k_j + k_l)] - \frac{\pi}{2} \sum_{\mu=1}^{M} [\mathrm{sgn}(k_j - \lambda_\mu) + \mathrm{sgn}(k_j + \lambda_\mu)] \\ 2J_\gamma &= \sum_{l=1}^{N} [\mathrm{sgn}(\lambda_\gamma - k_l) + \mathrm{sgn}(\lambda_\gamma + k_l)] - \sum_{\mu\neq\gamma}^{M} [\mathrm{sgn}(\lambda_\gamma - \lambda_\mu) + \mathrm{sgn}(\lambda_\gamma + \lambda_\mu)]. \end{aligned} \tag{16}$$

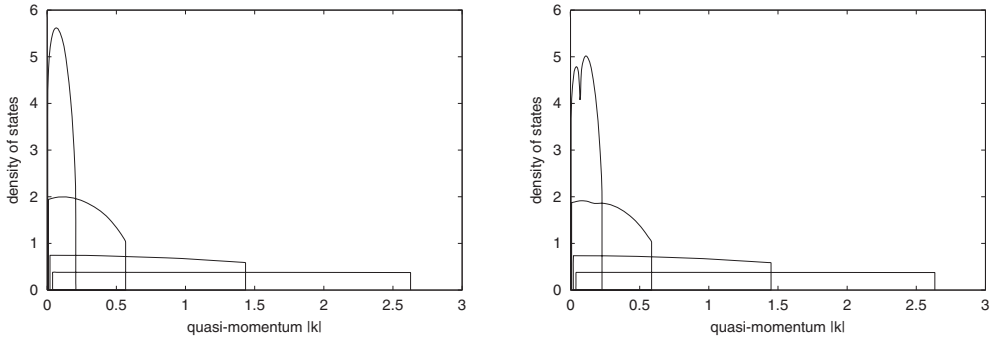

Figure 1. (Left) The density of state in $|k|$-space for the ground state. The distribution gradually changes from a histogram to a narrow peak for the coupling from strong to weak. (Right) The density of state in $|k|$-space in the presence of one isospin rapidity at $J_1 = 30$ where a rift evidently appeared. The figure is plotted for $N = L = 100$ and $c = 10, 1, 0.1, 0.01$.

We may choose both the subscripts of the J_j and J_γ in increasing order so that the second equation of equations (16) becomes

$$\sum_{l=1}^{N} \mathrm{sgn}(\lambda_\gamma - k_l) = 2J_\gamma + 2\gamma - N - 2 \tag{17}$$

which also gives rise to

$$\sum_{l=1}^{N} [\mathrm{sgn}(\lambda_{\gamma+1} - k_l) - \mathrm{sgn}(\lambda_\gamma - k_l)] = 2(J_{\gamma+1} - J_\gamma) + 2. \tag{18}$$

Thus, for $J_{\gamma+1} - J_\gamma = m$, there must exist exactly $m + 1$ momenta of k_l satisfying $\lambda_\gamma < k_l < \lambda_{\gamma+1}$.

Similarly, from the first equation of equations (16) we have

$$2(I_{j+1} - I_j) - \frac{2L}{\pi}(k_{j+1} - k_j) - 2 = -\sum_{\mu=1}^{M} [\mathrm{sgn}(k_{j+1} - \lambda_\mu) - \mathrm{sgn}(k_j - \lambda_\mu)]. \tag{19}$$

For $I_{j+1} - I_j = n$, there will be $k_{j+1} - k_j = (n + m - 1)\pi/L$ if there are m λ_γ satisfying $k_j < \lambda_\gamma < k_{j+1}$. Therefore, an isospin rapidity of value λ_μ always repels the quasi-momentum away from that value. The ground state for $c \to 0$ does not allow any possible existence of λ_μ between k_1 and k_N. Thus from equations (17) and (18) we can conclude that the total number of isospin down should be zero. The quantum numbers $\{I_j^0\}$ of the ground state are given by the successive positive numbers $I_1^0 = 1$, $I_{j+1}^0 - I_j^0 = 1$. In the thermodynamic limit, as the difference between two adjacent k is small, the contribution of any existing λ to the density of k-rapidity at points $k = \lambda$ will bring about a rift (see figure 1 right). Therefore, an existing λ_μ will suppress the density of state in k-space at the point $k = \lambda_\mu$. The more isospin rapidities there are, the higher the energy will be. In contrast to the spin-$\frac{1}{2}$ Fermi system whose ground state is $SU(2)$ singlet, the ground state of trapped two-component bosons is an isospin 'ferromagnetic' state which is described by the quantum number

$$\{I_j\} := \{1, \ldots, N\} \qquad \{J_\gamma\} = \text{empty}. \tag{20}$$

Since the existence of open boundary conditions makes the backwards scattering $k \to -k$ possible, there exists an additional term $\tan^{-1}(2k/c)$ which contributes a negative value to $\rho(k)$, especially $\rho(0)$ is obviously suppressed. This phenomenon is different from that for

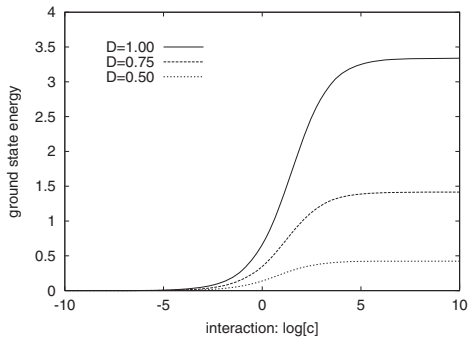

Figure 2. The ground state energy versus the coupling for different densities $D = 1.0, 0.75, 0.5$.

the system with periodic conditions. We interpret it as being due to the confinement by the infinite-depth well, for example, the ground state for a single particle is not $k = 0$ but $k = \pi/L$ in open boundary conditions. The density of states of the ground state for various coupling is plotted in figure 1 (left panel).

By introducing

$$\begin{aligned} L\rho\left(\frac{k_j + k_{j+1}}{2}\right) &= \frac{1}{k_{j+1} - k_j} \\ L\sigma\left(\frac{\lambda_\gamma + \lambda_{\gamma+1}}{2}\right) &= \frac{1}{\lambda_{\gamma+1} - \lambda_\gamma} \end{aligned} \tag{21}$$

the density corresponding to the quantum numbers (20) satisfies the integral equation

$$\rho_0(k) = \frac{1}{\pi} - \frac{1}{L}K_{1/2}(k) + \int_0^{Q_0} K_1(k|k')\rho_0(k')\,\mathrm{d}k' \tag{22}$$

where

$$K_n(x) = \frac{1}{\pi}\frac{nc}{n^2c^2 + x^2}$$

and

$$K_n(x|y) = K_n(x - y) + K_n(x + y)$$

ρ_0, Q_0 are, respectively, the density and integration limit for the ground state. The $K_{1/2}(k)$ term in (22) comes from the fact that $\tan^{-1}(\frac{k_j - k_l}{c}) \to 0$ but $\tan^{-1}(\frac{k_j + k_l}{c}) \to \tan^{-1}(\frac{k_j}{c/2})$ when taking account of the $l = j$ term in the thermodynamic limit process. The particle number per length is determined by

$$D = \frac{N}{L} = \int_0^{Q_0} \rho_0(k)\,\mathrm{d}k \tag{23}$$

equations (22) and (23) determine Q_0 and $\rho_0(k)$. Then the energy can be calculated using

$$\frac{E_0}{L} = \int_0^{Q_0} k^2\rho(k)\,\mathrm{d}k \tag{24}$$

which is explicitly $\frac{1}{3}\pi^2 D^3(1 - \frac{4}{c}D)$ in the strong coupling limit $c \gg 1$. In the general case one needs to solve the equations numerically. We show the ground state energy for various densities $D = 1.0, 0.75, 0.5$ in figure 2.

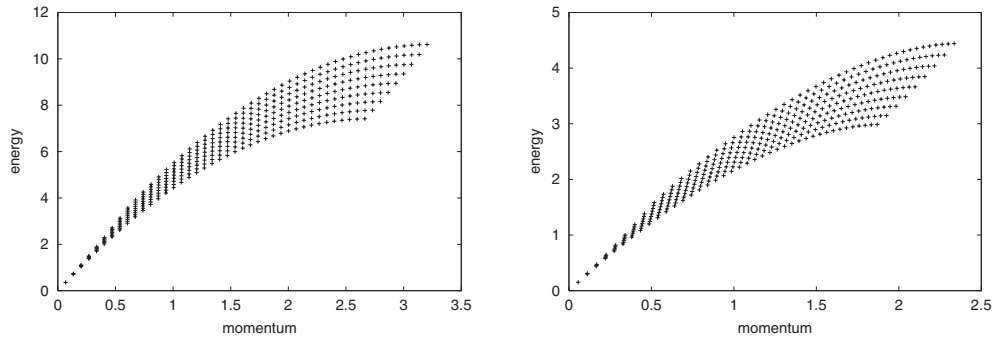

Figure 3. The holon–antiholon excitation spectrum calculated for $N = L = 40$ and $c = 10$ (left) and $c = 1$ (right).

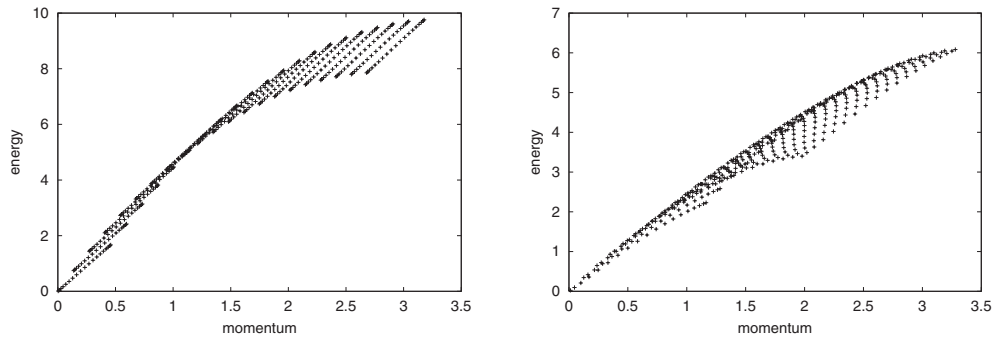

Figure 4. The holon–isospinon excitation spectrum calculated for $N = L = 20$, $c = 10$ (left) and $c = 1$ (right).

4. Low-lying excited states

The excited states are obtained by the variation of the configuration $\{I_j, J_\gamma\}$ from that of the ground state. The simplest case is to remove one I from the configuration of the ground state and add a new I_n outside the original sequence. We call this a holon–antiholon excitation which is described by

$$\{I_j\} = \{1, \ldots, n-1, n+1, \ldots, N, I_n\}$$

with $|I_n| > N$ and $M = 0$. To investigate the excited states, we shall consider systems of finite size first, then take thermodynamics limits. Although the total momentum $\sum_j k_j$ is no longer a constant in open boundary conditions, $\sum_j |k_j|$ is still a constant for a given energy eigenstate. We plotted the numerical results for the energy–'momentum' spectrum in figures 3 and 4 for $N = 40$, $L = 40$ where the x-axis represents the quantitive change $\sum_j (|k_j| - |k_j^0|)$. We notice that the overall structure of the spectrum does not change substantially between weak ($c = 1$) and strong ($c = 10$) coupling regimes. For a finite-size system, the holon–antiholon excitation gap opens, and its dependence on the inter-particle interaction is shown by the upper line of figure 5.

In the thermodynamic limit, it is plausible to calculate the excitation energy by making $\rho(k) = \rho_0(k) + \rho_1(k)/L$ where $\rho_0(k)$ is the density of the ground state. By creating a hole

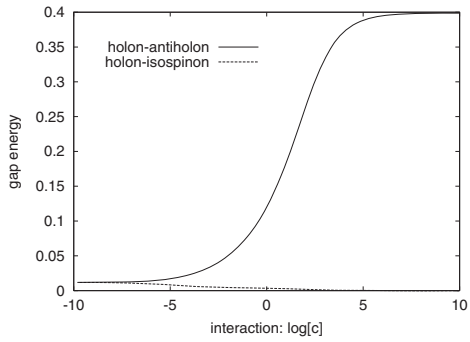

Figure 5. The finite-size energy gap versus interaction for $N = L = 50$, obviously $c = 0$ corresponds to $0.02 = 1/L$.

inside the quasi-Fermi sea $\bar{k} \in [0, Q_0]$ and an additional $k_p > Q_0$ outside it, we have

$$\rho_1(k) + \delta(k - \bar{k}) = \int_0^{Q_0} \mathrm{d}k' \, \rho_1(k') K_1(k|k') + K_1(k|k_p). \tag{25}$$

The excitation energy consists of two terms $\triangle E = \int k^2 \rho(k) \, \mathrm{d}k - \int k^2 \rho_0(k) \, \mathrm{d}k = \varepsilon_h(\bar{k}) + \varepsilon_a(k_p)$. The holon energy ε_h and antiholon energy $\varepsilon_a(k_p) = -\varepsilon_h(k_p)$ are given by

$$\begin{aligned} \varepsilon_h(\bar{k}) &= -\bar{k}^2 + \int_0^{Q_0} k^2 \rho_1^h(k, \bar{k}) \, \mathrm{d}k \\ \rho_1^h(k, \bar{k}) &= -K_1(k|\bar{k}) + \int_0^{Q_0} K_1(k|k') \rho_1^h(k'|\bar{k}). \end{aligned} \tag{26}$$

Another interesting excitation is to flip one isospin down, i.e. $M = 1$. As discussed above, a single λ can suppress the density of state at $k = \lambda$. We plot the density of the lowest energy state for various couplings in figure 1 (right panel) with $N = 100$, $L = 100$ and $J_1 = 30$. The holon–isospinon excitation is characterized by the Young tableau $[N - 1, 1]$, accordingly, I take N distinct integers and $I_1 < J_1 < I_N$, namely

$$\begin{aligned} I_1 &= -N/2 + \delta_{1,j_1} \qquad (1 \leqslant j_1 \leqslant N + 1) \\ I_j &= I_{j-1} + 1 + \delta_{j,j_1} \qquad (j = 2, \ldots, N) \end{aligned}$$

where $\delta_{\alpha,\beta}$ is the usual Kronecker symbol. The excitation spectra are plotted in figure 4, for $L = 20, N = 20, c = 10$ and $c = 1$ respectively. The lowest energy state of one isospin down is to make the absolute value of J_1 as large as possible in order to avoid the enhancement of energy caused by the suppression of the k-distribution. In the thermodynamic limit, it can be shown that this mode is gapless. For a finite particle system, however, it still has a gap of order $1/L$ which decreases more quickly than that for periodic conditions for particle number ranging from small to large. Figure 5 shows its dependence of the gap on the interaction. We presented in figure 6 the behaviour of the finite-size spin gap as a function of $1/L$ at $c = 1.0, 10.0, 100.0$.

In the thermodynamic limit, we should take into account the hole in the k-sector and the $\rho_1(k)$ should be solved from

$$\rho_1(k) + \delta(k - \bar{k}) = \int K_1(k|k') \rho_1(k') \, \mathrm{d}k' - K_{1/2}(k|\lambda). \tag{27}$$

One can find that the energy of holon–isospinon excitation consists of two terms $\triangle E = \varepsilon_h(\bar{k}) + \varepsilon_{\mathrm{iso}}(\lambda)$. The ε_h is determined by equations (26) and the $\varepsilon_{\mathrm{iso}}$ by $\varepsilon_{\mathrm{iso}}(\lambda) = \int k^2 \rho_1^{\mathrm{iso}}(k, \lambda) \, \mathrm{d}k$ with

$$\rho_1^{\mathrm{iso}}(k, \lambda) = -K_{1/2}(k|\lambda) + \int_0^{Q_0} K_1(k|k') \rho_1^{\mathrm{iso}}(k', \lambda) \, \mathrm{d}k'. \tag{28}$$

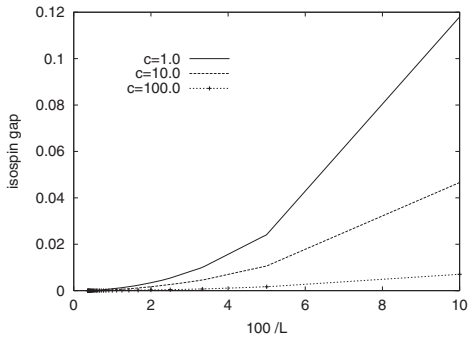

Figure 6. Finite-size effects: isospin gap as a function of $100/L$ for $D = N/L = 1$, plotted for various inter-particle interactions.

5. Thermodynamics at finite temperature

For the ground state (i.e. at zero temperature), the charge rapidity k are real roots of the Bethe-ansatz equations (14). For the excited state, however, the isospin rapidity can be complex roots [11] which always form a 'bound state' with several other λ. This arises from the consistency of both sides of the Bethe-ansatz equations [12]. The *n-string* of λ rapidity is defined as

$$\Lambda_a^{nj} = \lambda_a^n + (n + 1 - 2j)\mathrm{i}c/2 + \mathrm{O}(\exp(-\delta N)) \tag{29}$$

where $j = 1, 2 \ldots n$. The number of total isospin down is determined by

$$M = \sum_{n=1}^{\infty} n M_n \tag{30}$$

where M_n denotes the number of n-strings. Equations (15) become

$$\begin{aligned}
\pi I_j &= k_j L + \sum_{l\neq j}^{N}\left[\tan^{-1}\left(\frac{k_j - k_l}{c}\right) + \tan^{-1}\left(\frac{k_j + k_l}{c}\right)\right] \\
&\quad - \sum_{an}\left[\tan^{-1}\left(\frac{k_j - \lambda_a^n}{nc/2}\right) + \tan^{-1}\left(\frac{k_j + \lambda_a^n}{nc/2}\right)\right] \\
\pi J_a^n &= \tan^{-1}\left(\frac{\lambda_a^n}{nc/2}\right) + \sum_{l=1}^{N}\left[\tan^{-1}\left(\frac{\lambda_a^n - k_l}{nc/2}\right) + \tan^{-1}\left(\frac{\lambda_a^n + k_l}{nc/2}\right)\right] \\
&\quad - \sum_{mbt} A_{mnt}\left[\tan^{-1}\left(\frac{\lambda_a^n - \lambda_b^m}{tc/2}\right) + \tan^{-1}\left(\frac{\lambda_a^n + \lambda_b^m}{tc/2}\right)\right]
\end{aligned} \tag{31}$$

where

$$A_{nmt} = \begin{cases} 1 & \text{for} \quad t = m + n, |m - n|(\neq 0) \\ 2 & \text{for} \quad t = n + m - 2, \ldots, |n - m| + 2 \\ 0 & \text{otherwise} \end{cases}$$

and the quantum numbers $\{I_j, J_a^n\}$ label the state beyond the ground state. The densities correspond to charge rapidity and isospin rapidity on the real axis for which the 'omitted k, λ values' must be taken into account. By introducing the density of holes for charge $\rho^h(k)$ and isospin n-string $\sigma_n^h(\lambda)$, we have

$$\begin{aligned}
\rho + \rho^h &= \frac{1}{\pi} - \frac{1}{L}K_{1/2}(k) + \int K_1(k|k')\rho(k')\,\mathrm{d}k' - \sum_n \int K_{n/2}(k|\lambda)\sigma_n(\lambda)\,\mathrm{d}\lambda \\
\sigma_n + \sigma_n^h &= \frac{1}{L}K_{n/2}(\lambda) + \int K_{n/2}(\lambda|k)\rho(k)\,\mathrm{d}k - \sum_{mt} A_{mnt}\int K_{t/2}(\lambda|\lambda')\sigma_m(\lambda')\,\mathrm{d}\lambda'
\end{aligned} \tag{32}$$

where the integration limits are $[0, \infty]$. In terms of the distribution functions of charge and isospin rapidities, the energy per length has the form $E_k/L = \int k^2\rho(k)\,\mathrm{d}k$, the total number of down isospins is $M/L = \sum_n n \int \sigma_n(\lambda)\,\mathrm{d}\lambda$ and the particle density of the system is $D = N/L = \int \rho(k)\,\mathrm{d}k$.

If we consider the energy arising from the external field Ω which is the Rabi field in two-component BEC experiments, the internal energy of the system is

$$E/L = \int (k^2 - \Omega)\rho(k)\,\mathrm{d}k + \sum_n 2n\Omega \int \sigma_n\,\mathrm{d}\lambda. \tag{33}$$

For a given $\rho(k)$, $\rho^h(k)$, $\sigma_n(\lambda)$ and $\sigma_n^h(\lambda)$, the entropy is of the form [13]

$$\begin{aligned} S/L = &\int [(\rho+\rho^h)\ln(\rho+\rho^h) - \rho\ln\rho - \rho^h\ln\rho^h]\,\mathrm{d}k \\ &+ \sum_n \int [(\sigma_n+\sigma_n^h)\ln(\sigma_n+\sigma_n^h) - \sigma_n\ln\sigma_n - \sigma_n^h\ln\sigma_n^h]\,\mathrm{d}\lambda \end{aligned} \tag{34}$$

where the Boltzmann constant is set to unity.

At finite temperature, we should minimize the free energy $F = E - TS - \mu N$ where μ is the chemical potential and S is the entropy of the system. Making use of the relations derived from equations (32)

$$\begin{aligned} \delta\rho^h &= -\delta\rho + \int K_1(k|k')\delta\rho\,\mathrm{d}k' - \sum_n \int K_{n/2}(k|\lambda)\delta\sigma_n\,\mathrm{d}\lambda \\ \delta\sigma_n^h &= -\delta\sigma_n + \int K_{n/2}(\lambda|k)\delta\rho\,\mathrm{d}k - \sum_{mt} A_{mnt}\int K_{t/2}(\lambda|\lambda')\delta\sigma_m(\lambda')\,\mathrm{d}\lambda' \end{aligned} \tag{35}$$

we obtain the following conditions from the minimum condition $\delta F = 0$:

$$\begin{aligned} \epsilon(k) = k^2 - \Omega - \mu - T&\int K_1(k|k')\ln(1+\mathrm{e}^{-\epsilon(k')/T})\,\mathrm{d}k' \\ &-T\sum_n \int K_{n/2}(k|\lambda)\ln[1+\mathrm{e}^{-\zeta_n(\lambda)/T}]\,\mathrm{d}\lambda \\ \zeta_n(\lambda) = 2n\Omega + T&\int K_{n/2}(k|\lambda)\ln[1+\mathrm{e}^{-\epsilon(k)/T}]\,\mathrm{d}k \\ &+T\sum_{mt} A_{mnt}\int K_{t/2}(\lambda|\lambda')\ln[1+\mathrm{e}^{-\zeta_m(\lambda')/T}]\,\mathrm{d}\lambda' \end{aligned} \tag{36}$$

where we have written

$$\begin{aligned} \frac{\rho^h(k)}{\rho(k)} &= \mathrm{e}^{\epsilon(k)/T} \\ \frac{\sigma_n^h(\lambda)}{\sigma_n(\lambda)} &= \mathrm{e}^{\zeta_n(\lambda)/T}. \end{aligned} \tag{37}$$

When $T \to 0$, equations (36) become

$$\mathcal{E}_0(k) = k^2 - \Omega - \mu + \int K_1(k|k')\mathcal{E}_0(k')\,\mathrm{d}k' \tag{38}$$

where the 'ferromagnetic' ground state is under consideration. Clearly, the integral equation gives the solution for the dressed energy [17] from which the ground-state energy can be given in terms of $\mathcal{E}_0$

$$E_0/L = \frac{1}{\pi}\int_0^{Q_0} \mathcal{E}_0(k)\,\mathrm{d}k. \tag{39}$$

The solution of equation (38) defines the energy bands. And the Fermi surface is determined by

$$\mathcal{E}_0(k_F) = 0 \tag{40}$$

since the ground-state configuration corresponds to the case that all states of $\mathcal{E}_0(k) < 0$ are fully filled. The bare energy $\mathcal{E}_0^{(0)}$ is the zero-order term of equation (38)

$$\mathcal{E}_0^{(0)} = k^2 - \Omega - \mu \tag{41}$$

which is valid in the strong coupling limit

Equations (36) can be solved by iteration. The coupled equations (32) of density distribution of charge and isospin are then of Fredholm type:

$$\begin{aligned}
\rho(1+\mathrm{e}^{\epsilon/T}) &= \frac{1}{\pi} - \frac{1}{L}K_{1/2}(k) + \int K_1(k|k')\rho(k')\,\mathrm{d}k' - \sum_n \int K_{n/2}(k|\lambda)\sigma_n(\lambda)\,\mathrm{d}\lambda \\
\sigma_n(1+\mathrm{e}^{\zeta_n/T}) &= \frac{1}{L}K_{n/2}(\lambda) + \int K_{n/2}(\lambda|k)\rho(k)\,\mathrm{d}k - \sum_{mt} A_{mnt} \int K_{t/2}(\lambda|\lambda')\sigma_m(\lambda')\,\mathrm{d}\lambda'.
\end{aligned} \tag{42}$$

Finally, we obtain the Helmholtz free energy $F = E - TS$

$$F = \mu N - T\int \ln[1+\mathrm{e}^{-\epsilon/T}]\left[\frac{L}{\pi} - K_{1/2}(k)\right]\mathrm{d}k - T\sum_n \int \ln[1+\mathrm{e}^{-\zeta_n/T}]K_{n/2}(\lambda)\,\mathrm{d}\lambda \tag{43}$$

and the pressure

$$P = -\frac{\partial F}{\partial L} = \frac{T}{\pi}\int \ln[1+\mathrm{e}^{-\epsilon/T}]\,\mathrm{d}k \tag{44}$$

which is formally the same as Yang and Yang's expression but the equations determining ϵ and ζ are different. Consequently, the partition function is given by $Z = \mathrm{e}^{-F/T}$. The thermodynamic functions, such as partition function Z and free energy F, are of importance for a thermal system. Given either of them, one is, in principle, able to calculate all thermodynamic properties for the system. However, equations (36) are so complicated that it is impossible to obtain the explicit form of ϵ and ζ for a general case. Moreover, we can also obtain some plausible results for some special cases in the next section.

6. Special cases

In general, the free energy of our model should be calculated using formula (43), where $\epsilon(k)$ and $\zeta_n(\lambda)$ are determined from equations (36). Equations (36) can be solved by a numerical method via iteration. In this section, we will only discuss the case of strong coupling at low temperature.

When $c \to \infty$, $K_n(k|k') = 0$ and $K_n(k) = 0$, we then have

$$\epsilon = k^2 - \Omega - \mu \tag{45}$$

and the free energy becomes

$$F/L = \mu D - \frac{T}{\pi}\int \ln[1+\mathrm{e}^{-\epsilon/T}]\,\mathrm{d}k \tag{46}$$

which can be solved by integration by parts [18]:

$$F/L = \mu D - \frac{2}{\pi}\left(\frac{1}{3}\mu^{3/2} + \frac{T^2\pi^2}{24\mu^{1/2}}\right) \tag{47}$$

where the external field is set to zero.

We cannot derive the specific heat directly from the previously obtained free energy because the chemical potential is a function of temperature. From equations (32), the density of charge rapidity has the form

$$\rho = \frac{1}{\pi}\frac{1}{1+\mathrm{e}^{(k^2-\Omega-\mu)/T}}. \tag{48}$$

Clearly, at zero temperature, the quasi-Fermi surface is just the square root of the chemical potential, so we have $\mu_0 = \pi^2 D^2$ which denotes the chemical potential at zero temperature. At low temperature, however, it is determined by

$$D = \frac{1}{\pi}\int_0^\infty \frac{1}{1+\mathrm{e}^{(k^2-\mu)/T}}\,\mathrm{d}k \tag{49}$$

we have

$$\mu = \pi^2 D^2\left[1+\frac{\pi^2 T^2}{24\mu^2}\right]^{-2}. \tag{50}$$

The second term in the bracket is small at low temperature, so we can replace μ by μ_0. The equation becomes

$$\mu = \mu_0\left[1-\frac{\pi^2 T^2}{24\mu_0^2}\right]^{-2} \tag{51}$$

then the free energy becomes

$$F/L = \mu_0 D\left[1+\frac{\pi^2 T^2}{12\mu_0^2}\right] - \frac{2}{3\pi}\mu_0^{3/2}\left[1+\frac{\pi^2 T^2}{4\mu_0^2}\right]. \tag{52}$$

Since by thermodynamics $S = -\partial F/\partial T$ and $C_v = T\partial S/\partial T$, we find the specific heat at low temperature is Fermi-liquid like:

$$S = C_v = \frac{T}{6D}. \tag{53}$$

It is the same as the result of the one-component case, since for the strong coupling limit the isospin and charge are decoupled, the contribution of the isospinon to the free energy vanishes. In figure 6, the finite-size energy gap of holon–isospinon in the strong coupling limit tends to zero.

7. Conclusions

In this paper, we have solved a system of one-dimensional trapped $SU(2)$ bosons with δ-function interaction by means of the Bethe-ansatz method. On the basis of Bethe-ansatz equations we first discussed the ground state of the system and found that the ground state is an isospin 'ferromagnetic' state which differs greatly from the spin-$\frac{1}{2}$ fermion systems. We studied the low excitation states by both numerical and analytic methods. It was shown that there are three elementary excitation modes, and the holon–antiholon and holon–isospinon excitations are gapless for large systems. For a finite system, we not only plotted some excitation spectra but also plotted the dependence of the finite-size energy gap on the inter-particle interaction. The thermodynamics of the system was studied by using the thermodynamic Bethe ansatz [13]. For strong coupling we found that the system exhibits the Fermi-liquid behaviour, i.e. the specific heat is a linear function of T at low temperature.

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